USEFULNESS OF THREE-MOMENT X^2 AND t APPROXIMATIONS

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Summary. It is illustrated numerically that 3-moment x^2 and t distributions can be successfully adopted to obtain satisfactory approximations for the percentage points and the probability integrals of a number of non-normal distributions.

1. *Introduction*. Let Z be a random variable the distribution function of which is either unknown or, if known, is difficult to tabulate. The problem is how to approximate to the percentage points and the probability integral of Z given its first few moments. Approximations for the percentage points of Z can be readily obtained by fitting appropriate 4-moment curves of Pearson system from Johnson, Nixon, Amos and Pearson's (1963) tables (see Pearson, 1963 and Johnson et. al., 1963). However, these tables do not provide the answer if the value of the probability integral corresponding to a specified value of Z and not just a standard percentage point is required. For this reason, it has seemed worthwhile examining x^2 and t approximations having only three moments common with Z. Since extensive tables of x^2 and t distributions are readily available, this method of approximation, although slightly less accurate, overcomes the above difficulty with the 4-moment approximations.

Let μ'_1 and μ_2 denote the mean and variance of Z and $\beta'_1 = \mu_3^2/\mu_2^3$ and $\beta'_2 = \mu_4/\mu_2^2$ its third and fourth standard cumulants, respectively.

2·1. THREE-MOMENT x^2-APPROXIMATIONS

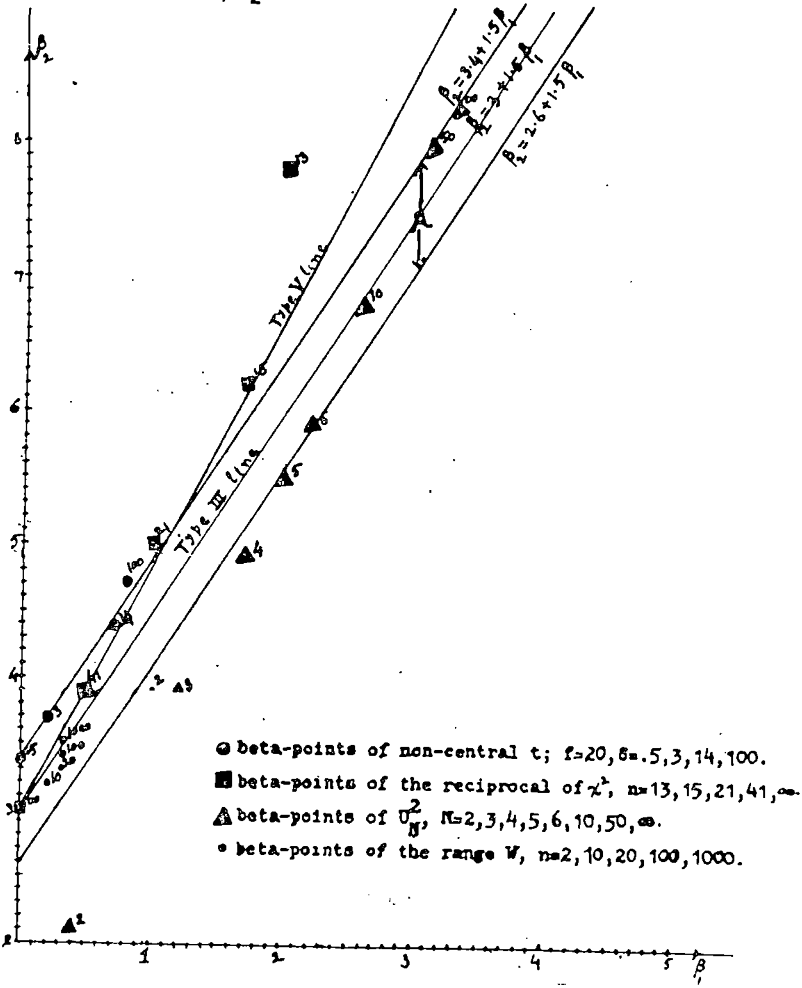
In the first place, suppose β'_1 deviates appreciably from zero. Write $X = \frac{1}{2}x^2$, where x^2 is a chi-square variate with degrees of freedom $\nu = 2m$. Suppose

$$X \sim (Z + c)/\rho \qquad \ldots(1)$$

and the parameters ρ, m and c are determined so that the right hand side of (1) has first three moments common with X. Equating these moments we obtain

$$\left.\begin{aligned} \rho &= \tfrac{1}{2}\mu_3/\mu_2 \\ m &= 4/\beta'_1 \\ \text{and} \qquad c &= m\rho - \mu'_1 \end{aligned}\right\} \qquad \ldots(2)$$

Fig.1. (β_1, β_2) points of various distributions.



As long as β'_2 does not deviate from the chi-square beta-value $\beta_2=3+6/m$, by more than say 0·4, the 3-moment x^2-distribution (2) may provide satisfactory approximations for the percentage points as well as the probability integral of Z, except perhaps for the extreme tails. To illustrate this numerically the following distributions

(*a*) non-central *t*-distribution,

(*b*) distributions of the range in normal samples,

(*c*) distribution of the mean deviation in normal samples,

(*d*) beta-distribution, and

(*e*) the distribution of the reciprocal of x^2 variate,

were considered, their percentage points or the probability integrals were evaluated from the x^2-approximation (2) and compared with the true values, the agreement was excellent except for the very extreme tails. We present these results in §§ 2·2—2·4, in case of the distributions (*a*), (*b*) and (*c*) only.

Apart from the above distributions there will certainly be several other distributions for which the x^2-distribution (2) will provide accurate approximations. For example the non-central x^2-distribution and the extremely non-normal distributions of goodness-of-fit statistics U^2_N and W^2_N. The corresponding 3-moment x^2-approximations were derived by Pearson (1959) and Tiku (1965a).

The beta-points of a few distributions are plotted in Fig. 1. It is interesting to note that although most of these points lie far away from the normal point ($\beta_1=0$, $\beta_2=3$), a few points lie close to the Type III line, within the region A (Fig. 1) of the (β_1, β_2) plane bounded by the lines $\beta_2=3{\cdot}4+1{\cdot}5\beta_1$ and $\beta_2=2{\cdot}6+1{\cdot}5\beta_1$. It seems that for the distributions, whose beta-points lie in the region A, the x^2-distribution (2) will generally provide reasonable approximations, except perhaps for the extreme tails.

2·2. Non-Central *t* Distribution

The non-central *t* distribution having f degrees of freedom and non-centrality parameter δ is given by—

$$f(t)=\frac{\Gamma(f+1)}{2^{\frac{1}{2}(f-1)}\,\Gamma(\frac{1}{2}f)\sqrt{\pi f}}\left(\frac{f}{f+t^2}\right)^{\frac{1}{2}(f+1)}$$

$$\cdot\exp\left[-\tfrac{1}{2}\left(\frac{f\delta^2}{f+t^2}\right)\right]Hh_f\left(\frac{-\delta t}{\sqrt{f+t^2}}\right) \qquad \ldots(3)$$

where

$$Hh_f(y) = \int_0^\infty \frac{u^f}{\Gamma(f+1)}\, e^{-\frac{1}{2}(u+y)^2}\, du$$

Resnikoff and Lieberman (1959) tabulated the percentage points and the probability integral of (3) for $f=1\ (1)24(5)49$. Hogben, Pinkham and Wilk (1961) derived a neat expression for the moments of t and give a table to facilitate the computation of the first four moments. Merrington and Pearson (1958) illustrated numerically that the distribution (3) is very closely represented by a Pearson Type IV curve at least for $0 \leqslant \delta \leqslant 3{\cdot}09\sqrt{f+1}$. For this range of δ, the beta-points (β_1, β_2) of (3) lie within the region A of Fig. 1, for $f \geqslant 20$.

The 100α percentage points for the non-central t were evaluated from the x^2-approximation (2), for the values of δ and f considerep by Merrington and Pearson (1958), interpolating linearly in Biometrika Table 8. These values were compared with Merington and Pearson's (Table 1) true values for $\alpha = 0{\cdot}01, 0{\cdot}05, 0{\cdot}95, 0{\cdot}99, 0{\cdot}995$

TABLE 1A

Values of 100α percentage points of non-central t distribution

f	δ	$m = \frac{1}{2}v$	$(\beta'_2 - \beta_2)$	Error in approx. (2) : (approx.—true)10^2					
				$\alpha=0{\cdot}01$	0·05	0·50	0·95	0·99	0·995
17	2·862	14·94	·45	15	2	0*	4	—4	—10
20	3·091	18·65	—·36	12	2	0	2	—4	—10
20	14·161	5·71	—·38	27	6	1	7	—5	—15
22	8·396	7·94	—·33	15	3	0	4	—3	—9
34	3·990	36·36	—·18	6	0	0	0	—3	—7
36	1·8	160	—·18	4	0	0	0	—3	—6
44	8·597	22·22	—·13	6	0	0	0	—3	—6
49	4·769	55·56	—·12	3	1	0	0	—3	—5
49	21·85	16	—·12	11	3	0	0	—4	—7
60	5·268	69·32	—·10	3	0	0	0	—2	—3
60	24·135	20·20	—·10	6	0	0	0	—3	—6

*Percentage points of $t/\sqrt{f}$

and with Resnikoff and Lieberman's (1957) true values of the percentage points of $t/\sqrt{f}$ for $\alpha=0{\cdot}50$. For $f=60$, these values were compared with the values obtained from 4-moment Pearson Type IV curves. The results are given in Table 1A. The x^2-distribution (2) gives satisfactory approximations, at any rate, for $0{\cdot}05\leqslant\alpha\leqslant0{\cdot}99$.

The approximate values of the probability integral of the distribution (3) evaluated from (2) are given in Table 1B, for $f=24$. These values were obtained by two-way interpolation in Biometrika Table 7, linear for X and linear for $m=\frac{1}{2}v$. It is clear the approximation (2) does not involve serious errors.

The x^2-approximation (2) should therefore prove useful as an approximation for the combination of values of δ and f not covered by Resnikoff and Lieberman's tables, at least for $0\leqslant\delta\leqslant3{\cdot}09\sqrt{f+1}$ and $f\geqslant20$, for approximating to the percentage points as well as the probability integeral of the non-central t distribution.

TABLE 1B

Approximate values of the Prob. $[t\leqslant t_0], f=24$

t_0 \ δ	11·632	13·260	14·035	15·451
12 247	0·591 (2)	0·290 (7)	0·177 (5)	0·040 (—13)
14·247	0·953 (—5)	0·632 (1)	0·502 (5)	0·274 (7)
16·166	0·960 (—2)	0·857 (—4)	0·776 (—3)	0·809 (—2)
18·126	0·991 (0)	0·955 (—2)	0·919 (—3)	0·809 (—3)
20 086	0·998 (0)	0·988 (0)	0·975 (—1)	0·927 (—3)
22·045	1·000 (0)	0·997 (0)	0·993 (0)	0·976 (0)

The figures in parentheses are the errors in units of third decimal place.

2·3. Distribution of the Range in Normal Samples

If x_1 and x_n are the lowest and highest values in a random sample of size n from the normal population then the probability integral of the range

$$W=(x_n-x_1)/\sigma \qquad \ldots(4)$$

may be expressed in the form (see Biometrika Tables, p. 43)

$$P(W/n)=n\int_{-\infty}^{\infty} f(x)\left\{\int_{x}^{x+w} f(u)\, du\right\}^{n-1} dX \qquad \ldots(5)$$

TABLE 2A *Comparison of 100α percentage points of the range, W*

n	$m=\frac{1}{2}\nu$	$(\beta'_2-\beta_2)$	Error in approx. (2) : (approx.—true)10^2								
			α=0·005	0·01	0·05	0·10	0·90	0·95	0·99	0·995	0·999
7	22·96	·09	−3	−2	0	0	0	0	1	2	5
10	25·32	·04	−1	−1	0	0	0	0	0	1	3
15	25·51	·01	1	1	0	0	0	0	0	0	0
20	24·59	—·02	1	1	0	0	0	0	0	0	0
60	19·90	—·05	1	1	0	0	0	0	0	—1	*
200	15·87	—·06	1	1	0	0	0	0	0	—1	*
1000	12·95	—·08	1	1	0	0	0	0	0	—2	*

*Lower and upper 0·1% points are not available in Johnson et al tables.

TABLE 2B. *Approximate values of the Prob.* $[W \leq W_0]$, *calculated from* (2)

W_0 \ n	4	6	8	10	15	20
1·20	0·1622 (−66)	0·0424 (7)	0·0107 (9)	0·0025 (3)	0·0001 (1)	0·0000 (0)
2·20	0·5965 (8)	0·3696 (—24)	0·2223 (—9)	0·1304 (—3)	0·0323 (0)	0·0074 (—2)
3·20	0·8962 (31)	0·7917 (12)	0·6857 (1)	0·5861 (—3)	0·3820 (6)	0·2404 (8)
4·20	0·9837 (−5)	0·9645 (—2)	0·9403 (−1)	0·9123 (—3)	0·8337 (−10)	0·7507 (—13)
5·20	0·9982 (—4)	0·9964 (—4)	0·9939 (—3)	0·9908 (—3)	0·9809 (0)	0·9677 (—4)
6·20	0·9999 (0)	0·9998 (0)	0·9997 (0)	0·9995 (0)	0·9988 (−1)	0·9980 (0)

The figures in parentheses are the errors in units of fourth decimal place.

TABLE 3A. *Comparison of the 100α percentage points of m**

n	$m=\frac{1}{2}v$	$(\beta'_2-\beta_2)$	*Error in approx. (2) : (approx.—true)10^3*								
			α=0·005	0·01	0 05	0·10	0·90	0 95	0·99	0·995	0·999
3	9·592	·34	*	—53	—8	5	—11	—7	18	33	77
5	17·391	·26	—20	—8	+2	4	—3	—3	5	10	26
7	25·478	·10	—10	—6	1	2	—2	—2	3	6	14

*gives negative value for m^*.

Table 3B. *Values of the 100α percentage points of m*, calculated from approximation (2)*

n	α= 0·005	0·01	0·05	0·10	0·90	0·95	0·99	0·995
10	0·327 (—6) [—64]	0·362 (—4) [—50]	0·464 (0) [—19]	0 522 (1) [—7]	1·005 (—2) [—7]	1·085 (—1) [—17]	1·242 (2) [—42]	1·302 (3) [—54]
15	0·411 (—2) [—41]	0.442 (—1) [—33]	0·529 (1) [—12]	0·578 (1) [—5]	0·973 (—1) [—5]	1·037 (—1) [—11]	1·159 (1) [—27]	1·206 (1) [—36]
30	0·521 (—1) [—20]	0·545 (0) [—16]	0·610 (0) [—6]	0·646 (0) [—2]	0·927 (0) [—2]	0·970 (0) [—5]	1·054 (0) [—14]	1·086 (1) [—18]

Values in round and square brackets are the errors, in units of third decimal place, involved in approximation (2) and normal approximation, respectively.

where

$$f(x)=\frac{1}{\sqrt{2\pi}}\ e^{-\frac{1}{2}x^2}.$$

The values of the first four moments of W are given in Biometrika Tables for $n=2$ (1) 20, 60, 100, 200, 500 and 1000. The beta-points of W lie exclusively within the region A of Fig. 1.

The values of the percentage points and the probability integral of W were evaluated from the approximation (2) and compared with the true values (Biometrika Tables 22 and 23) for $n\leqslant 20$ and with the values obtained by fitting appropriate 4-moment Pearson curves for $n>20$. The results are given in Table 2A and Table 2B. The accuracy of x^2-approximation (2) is remarkable and should be useful for approximating to the distribution (5), for $n>20$, the values of n not covered in Biometrika Tables.

2·4. Distribution of the Mean Deviation

The distribution of the mean deviation

$$m^*=\frac{1}{\sigma}\sum_{i=1}^{n}\left|x_i-\bar{x}\right|/n \quad \ldots(6)$$

calculated from a random sample of size n from a normal population does not assume a simple functional form. The values of the first four moments of m^* for $n=2(1)$ 20, 30, 60 are given in Biometrika Tables (Table 20). The probability integral of m^* was tabled by Hartley (1945) for $n=2(1)10$. The percentage points of m^* are given in Biometrika Tables (Table 21) for $n\leqslant 10$. The values of the percentage points of m^* were evaluated from the 3-moment x^2—approximation (2) and compared with the true values for $n\leqslant 10$ and with the values obtained from Pearson curves, having the correct first four moments, for $n>10$. The results are given in Tables 3A and 3B.

It may be noted that for $n\geqslant 10$, the beta-points of m^* assume values very close to the normal values ($\beta_1=0$, $\beta_2=3$), [see Biometrika Tables, p. 164], but even here, the x^2-approximation (2) gives results quite a good deal better than the Normal approximation.

Note that the beta-points of m^* lie very close to the Type III line, for $n\geqslant 3$.

3·1. Three-moment t approximations

Three moment x^2 approximations are useful in case β'_1 is different from zero. But if β'_1 is either zero, or very nearly so, 3-moment t—approximations for the distribution of Z can be obtained as follows :

Suppose,

$$t \cong (Z+d)/h \qquad \ldots(7)$$

where t is distributed as Student's t having f degrees of freedom and f, d and h are determined so that the right hand side of (7) has first, second and fourth moments common with t. Equating these moments, we obtain

$$\left.\begin{aligned} f &= 4\,(\beta'_2-1{\cdot}5)/(\beta'_2-3)\ ,\ \beta'_2>3 \\ h &= \sqrt{\mu_2\,(1-2/f)} \\ \text{and}\qquad d &= -\mu'_1 \end{aligned}\right\} \qquad \ldots(8)$$

For 'leptokurtic' distributions which are symmetrical, or very nearly so, *i.e.*, β'_1 is less than say 0·02, the 3-moment t-distribution (8) may provide reasonable approximations. To illustrate this numerically in §§ 3·2 and 3·3, the values of the probability integrals of the distributions of Fisher's (1921)

$$Z=\frac{1}{2}\log_e\frac{1+r}{1-r}$$

and Barton, David and O'Neill's (1960)

$$Z^*=\frac{1}{2}\log_e F^*$$

are calculated from (8) and compared with the true values. The agreement is excellent.

3·2. Distribution of $Z=\frac{1}{2}\log\frac{1+r}{1-r}$.

Fisher (1921) difined the statistics

$$Z=\frac{1}{2}\log_e\frac{1+r}{1-r}=\tan h^{-1}{}_r \qquad \ldots(9)$$

where r is the correlation coefficient calculated from a random sample of size n from a bivariate normal population. First four moments, derived by Fisher (1921) and corrected by Gayen (1951), are reproduced in Biometrika Tables [equation (48), p. 29].

The values of the probability integral of Z for $n=11$ and $\rho=0{\cdot}8$ (the population correlation coefficient) were evaluated from the t-distribution (8) by two-way linear interpolation in Biometrika Table 9. The values of the t-distribution (8) in this case are

$$f=28{\cdot}9480,\ h=0{\cdot}3336 \text{ and } d=-1{\cdot}1442.$$

These values were compared with Fisher's (1921) Normal approximation $N(\tan h^{-1}\rho,\ 1/\sqrt{(n-3)})$ and Gayen's (1951) Normal approximation $N(\tan h^{-1}\rho+\rho/2(n-1), \sqrt{\{[l/(n-1)]+[(4-\rho^2)/2(n-1)^2]\}}$. The results are given in Table 4.

It is clear, the t-approximation (8) is much more effective than the Normal approximations.

Note that $\sqrt{\beta'_1}(Z)=0\cdot0016$, for $n=11$ and $\rho=0\cdot8$.

For large samples it will suffice to take

$$\beta'_2=3+\frac{2}{n-3}$$

and

$$\mu_2=\frac{1}{n-3} \text{ in (8).}$$

TABLE 4

Comparison of approximate and true values of the probability integral of Z

Z	t	Error in approx. (10) : (approx−true)10^5			True values (Gayen 1951)
		Normal approx. Fisher	Normal approx. (Gayen)	Approx. (8)	
0·0000	−3·4300	−25*	−46	4	0·00089
0·1003	−3·1293	−28	−75	7	0·00194
0·2027	−2·8224	−13	−110	11	0·00419
0·3095	−2·5022	51	−137	16	0·00900
0·4236	−2·1602	218	−118	30	0·01940
0·5493	−1·7834	555	9	35	0·04223
0·6931	−1·3523	1073	310	32	0·09311
0·7753	−1·1059	1335	516	28	0·13872
0·8673	−0·8302	1485	706	12	0·20658
0·9730	−0·5133	1374	798	−8	0·30596
1·0986	−0·1368	853	698	−35	0·44643
1·1723	0·0841	445	575	−36	0·53348
1·2562	0·3356	3	435	−3	0·63020
1·3540	0·6288	−360	332	28	0·73243
1·4722	0·9831	−495	324	79	0·83226
1·6226	1·4339	−315	399	111	0·91764
1·8318	2·0610	15	384	77	0·97489
2·1847	3·1189	70	114	17	0·99776
Mean absolute error :		0·00484	0·00338	0·00032	

*Borrowed from Gayen (1951).

TABLE 5. *Comparison of the probabilities,* $\rho[Z^* \surd Z^*\alpha]$

	$\lambda=1, f_1=f_2=10$				$\lambda=5, f_1=f_2=4$				$\lambda=3, f_1=f_2=2$			
		(approx.—true)10^4				*(approx.—true)10*^{4}				*(approx.—true)10*^{4}		
α	$Z^*\alpha$	*Approx.* (8)	*Normal approx*	*True*	$Z^*\alpha$	*Approx.* (8)	*Normal approx.*	*True*	$Z^*\alpha$	*Approx.* (8)	*Normal approx.*	*True*
0·005	0·8829	0	11	0·9929	1·5711	21	45	0·9794	2·6467	23	62	0·9876
·01	·7894	−1	11	·9861	1·3856	26	28	·9604	2·2976	35	64	·9752
·025	·6564	0	3	·9662	1·1311	19	43	·9086	1·8319	35	−26	·9391
·05	·5457	−1	−12	·9342	·9272	−6	−121	·8332	1·4723	−2	−154	·8814
·10	·4213	−1	−35	·8730	·7064	−53	−161	·7077	1·0987	−81	−285	·7746
·25	·2196	−2	−43	·7015	·3624	−67	−26	·4420	·5493	−113	−118	·5155
·50	·0000	0	14	·4411	·0000	4	123	·1880	·0000	36	238	·2362
·75	−·2196	2	46	·2056	−·3624	30	59	·0573	−·5493	47	162	·0812
·90	−·4213	1	18	·0770	−·7064	15	−8	·0154	−1·0987	14	−3	·0259
·95	−·5457	0	−2	·0371	−·9272	8	−17	·0064	−1·4723	3	−37	·0120
·975	−·6564	0	−11	·0180	−1·1311	5	−14	·0028	−1·8319	2	−34	·0058
·99	−·7894	0	−12	·0070	−1·3856	2	−7	·0010	−2·2976	1	−19	·0022
·995	−·8829	0	−9	·0034	−1·5711	1	−4	·0005	−2·6467	0	−10	·0011
Mean absolute error :		0 0001	0·0017			0·0020	0·0050			0·0030	0·0093	

3·3 Distribution of $Z^*=\frac{1}{2}\log_e F^*$

Barton et al. studied the distribution of the transformed variate

$$Z^*=\frac{1}{2}\log_e F^*,$$

where F^* is a non-central F variate having (f_1, f_2) degrees of freedom. They evaluated expressions for the first four moments of Z^* and considered the possibility of approximating its distribution by a Normal distribution in case λ, the non-centrality parameter, is small and f_1 and f_2 are not very different. The values of the probability integral of Z^* were evaluated from approximation (8), by two-way linear interpolation in Biometrika Table 9, for

$$\lambda=1, f_1=f_2=10;\ \lambda=5,\ f_1=f_2=4$$

and

$$\lambda=3, f_1=f_2=2.$$

The beta-values of Z^* and the parameters of the distribution (8) for the above combinations of λ, f_1 and f_2 are as follows :

λ	f_1	f_2	β_1	β_2	f	h	d
1	10	10	0·0003	3·2202	31·2480	0·3113	−0·0480
5	4	4	0 0985	3·7025	12·5409	0·4830	−0·4428
3	2	2	0·1165	4·4884	8 0312	0·7290	−0·5414

The values were compared with the values obtained from Barton, David and O'Neill's Normal distribution. The results are given in Table 5. The true values of the Prob. $(Z^*\leq{}^*_a)$ were evaluated from Tiku's (1965*b*) equation (7·2). It is clear, the 3-moment *t*-distribution (8) provides better approximations than the Normal distribution.

Similar 3-moment *F*-distributions, which are useful in approximating to the non-central and doubly non-central *F* distributions, are given by Tiku (1965 *b*, *C*).

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